CFT and gravity : exercises week 3.

$\boxed{\text{Exercise 1}}$ See Mathematica notebook

$\boxed{\text{Exercise 2}}$

⊗ <u>4 different operators</u> :

The prefactor can only be a function of distances $|x_{ij}|$ (by Lorentz invariance) and it is fixed by inversion:

Under $x^{\mu'} = \dfrac{x^\mu}{x^2}$:
$$x'^2_{ij} = \frac{x^2_{ij}}{x_i^2\, x_j^2} \qquad (1)$$

(This is proven in ex. 2.54)

On the other hand, under inversion a scalar operator transforms as:

$$\mathcal{O}'(x) = (x^2)^{-\Delta}\, \mathcal{O}(x')$$

In a correlation functions, operators are dummy variables, so:

$$G(x_1,\dots x_4) = \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\,\mathcal{O}_4(x_4)\rangle = \langle \mathcal{O}'(x_1)\ \dots\ \mathcal{O}'(x_4)\rangle$$

$$= (x_1^2)^{-\Delta_1}\,(x_2^2)^{-\Delta_2}\,(x_3^2)^{-\Delta_3}\,(x_4^2)^{-\Delta_4}\,\langle \mathcal{O}(x_1')\dots \mathcal{O}(x_4')\rangle$$

$$= (x_1^2)^{-\Delta_1}\dots(x_4^2)^{-\Delta_4}\;G(x_1',\dots,x_4') \qquad (2)$$

Putting (1) and (2) together we find a simple rule:
" Assign to x_{ij}^2 degree 1 in x_i and x_j. Then x_i must appear with overall degree $-\Delta_i$ ".

In equations:

$$G(x_1,\dots x_4) = x_{12}^{2\,\alpha_{12}}\, x_{13}^{2\,\alpha_{13}}\, x_{14}^{2\,\alpha_{14}}\, x_{23}^{2\,\alpha_{23}}\, x_{24}^{2\,\alpha_{24}}\, x_{34}^{2\,\alpha_{34}}\; A(u,v)$$

$$\begin{cases} \alpha_{12} + \alpha_{13} + \alpha_{14} = -\Delta_1 \\ \alpha_{12} + \alpha_{23} + \alpha_{24} = -\Delta_2 \\ \alpha_{13} + \alpha_{23} + \alpha_{34} = -\Delta_3 \\ \alpha_{14} + \alpha_{24} + \alpha_{34} = -\Delta_4 \end{cases}$$

The system is under-determined because we could factor out powers of the cross-ratios u and v.
One convenient choice is to use u to set: $\alpha_{12} = -\frac{\Delta_1+\Delta_2}{2}$.
Then we can still use v, and set: $\alpha_{23} = 0$

Then the solution is unique:

$$\alpha_{34} = -\frac{\Delta_{34}}{2} \qquad \alpha_{13} = -\frac{\Delta_{34}}{2} \qquad \alpha_{24} = \frac{\Delta_{12}}{2}$$

$$\alpha_{14} = -\frac{\Delta_{12}}{2} + \frac{\Delta_{34}}{2}$$

All together:

$$G(x_1, \dots x_4) = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\Delta_{12}/2} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\Delta_{34}/2} \frac{A(u,v)}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2}{2}} \left(x_{34}^2\right)^{\frac{\Delta_3+\Delta_4}{2}}}$$

⊗ Counting cross-ratios

We can always choose 3 points at positions $0, 1, \infty$ in direction 1, say.
Then we cannot change anymore the first component of the other position vectors: we will get $n-3$ cross-ratios from this.
Now consider the $(d-1)$-dimensional subspace orthogonal to direction 1. We can use a rotation to align an axis to the point x_4 (projected on the subspace).
In other words, $x_1 \dots x_4$ now have components:

$$\begin{cases} x_1 = (0, \dots, 0) \\ x_2 = (1, 0 \dots 0) \\ x_3 = (\infty) \\ x_4 = (u_{41}, u_{42}, 0 \dots 0) \end{cases}$$

We keep going and set the last $d-2$ components of x_5 to zero etc.

The operation ends either because we run out of dimensions or of points:

<u>CASE I</u> $\quad d-1 \geq n-3 \quad \rightsquigarrow \quad \boxed{d \geq n-2}$

In this case the components of the $n-3$ points form the matrix:

$$d\left\{\underbrace{\begin{pmatrix} u_{41} & u_{51} & & u_{n1} \\ u_{42} & u_{52} & \cdots & u_{n2} \\ 0 & u_{53} & & \vdots \\ \vdots & 0 & & \\ & \vdots & & u_{n\,n-2} \\ & & & 0 \\ & & & \vdots \\ 0 & 0 & & 0 \end{pmatrix}}_{n-3}\right.$$

the number of parameters (= cross-ratios) is:

$$\#_{c.r.} = n-3 + \frac{(n-3)(n-2)}{2} = \boxed{\frac{n(n-3)}{2}}$$

<u>CASE 2</u> $\quad d-1 < n-3 \quad \rightsquigarrow \quad \boxed{d < n-2}$

The matrix now is

$$d\left\{\underbrace{\begin{pmatrix} u_{41} & \cdots & u_{d+2,1} & u_{d+3,1} & & u_{n1} \\ u_{42} & & \vdots & \vdots & & \vdots \\ 0 & & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & u_{d+2,d} & u_{d+3,d} & & u_{nd} \end{pmatrix}}_{n-3}\right.$$

And

$$\#_{c.r.} = n-3 + \frac{d(d-1)}{2} + (d-1)(n-d-2)$$
$$= \boxed{nd - \frac{(d+2)(d+1)}{2}}$$

The last formula can be also found as follows:

$$\begin{cases} nd = \# \text{ of coordinates specifying } n \text{ points} \\ \dfrac{(d+2)(d+1)}{2} = \# \text{ generators of the conformal group} \end{cases}$$

This seems general. Why does it fail in case I?
It fails because when d is large some generators do not act anymore: once we place n points in a $(n-2)$-dimensional subspace, rotations in the orthogonal $(d-n+2)$-dim supspace act trivially. There are $\frac{(d-n+2)(d-n+1)}{2}$ such rotations, so the counting is:

$$\#_{cr}^{\text{CASE I}} = nd - \frac{(d+2)(d+1)}{2} + \frac{(d-n+2)(d-n+1)}{2}$$

$$= \frac{n(n-3)}{2} \quad \checkmark$$

OBSERVATION: counting cross-ratios becomes trivial in "embedding space", a space you will learn all about shortly.

Exercise 3

1) $x_1 \leftrightarrow x_2$: sends $\left(u \to \frac{u}{v},\ v \to \frac{1}{v}\right)$ and leaves the prefactor invariant:

$$\Rightarrow \quad A(u,v) = A\left(\frac{u}{v}; \frac{1}{v}\right)$$

2) $x_1 \leftrightarrow x_3$: • $u \leftrightarrow v$

• $x_{12}^2 x_{34}^2 \to x_{14}^2 x_{23}^2 = x_{12}^2 x_{34}^2 \frac{v}{u}$

$$\Rightarrow \quad A(u,v) = \left(\frac{u}{v}\right)^{\Delta} A(v,u)$$

3) $x_1 \leftrightarrow x_4$: × $u \to \frac{1}{u}$, × $v \to \frac{v}{u}$

× $x_{12}^2 x_{34}^2 \to x_{12}^2 x_{34}^2 \frac{1}{u}$

$$\Rightarrow \quad A(u,v) = u^{\Delta} A\left(\frac{1}{u}, \frac{v}{u}\right)$$

But this is automatically satisfied given the first two:

$$u^{\Delta} A\left(\frac{1}{u}, \frac{v}{u}\right) \underset{(1)}{=} u^{\Delta} A\left(\frac{1}{v}, \frac{u}{v}\right) \underset{(2)}{=}$$

$$\underset{(2)}{=} u^{\Delta}\left(\frac{1/v}{u/v}\right)^{\Delta} A\left(\frac{u}{v}, \frac{1}{v}\right) = A\left(\frac{u}{v}, \frac{1}{v}\right)$$

$$\underset{(1)}{=} A(u, v) \quad \checkmark$$

4) All others permutations are generated by these, so this is the full set.

$\boxed{\text{Exercise 4}}$ Most of the solution is in the Mathematica notebook. Let's just compute N here: point c)

Start from the Green equation:

$$\langle \varphi(x)\varphi(y)\rangle = G(x-y), \qquad \Box G(x) = -\delta^d(x)$$

Go to Fourier space:

$$G(x) = \int \frac{d^dp}{(2\pi)^d} \tilde{G}(p)\, e^{+ip\cdot x}$$

$$\delta^d(x) = \int \frac{d^dp}{(2\pi)^d} e^{ip\cdot x}$$

$$\to \quad \tilde{G}(p) = \frac{1}{p^2}$$

$$\to \quad G(x) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} e^{ip\cdot x}$$

To do the Fourier transform, we repeat the steps of ex. 1.1.2 (First week):

$$\frac{1}{p^2} = \int_0^\infty dt \; e^{-tp^2} \quad + \text{ Gaussian integration}$$

$$G(x) = \int_0^\infty dt \; \frac{1}{(4\pi t)^{d/2}} \; e^{-x^2/4t}$$

Then change variables $z = \frac{x^2}{4t}$

$$G(x) = \left(\frac{x^2}{4}\right)^{1-d/2} \frac{1}{(4\pi)^{d/2}} \int_0^\infty dz \; z^{-2+d/2} \; e^{-z} =$$

$$= \frac{\Gamma\left(\frac{d}{2}-1\right) / 4\pi^{d/2}}{(x^2)^{\frac{d}{2}-1}} = \frac{N}{(x^2)^{d/2-1}}$$